The University of the State of New York

THE STATE EDUCATION DEPARTMENT

EXAMINATION IN EXPERIMENTAL TWELFTH YEAR MATHEMATICS

June 1967

The last page of the booklet is the answer sheet, which is perforated. Fold the last page along the perforation and then, slowly and carefully, tear off the answer sheet. Now fill in the heading of your answer sheet. When you have finished the heading, you may begin the examination immediately.

Part I

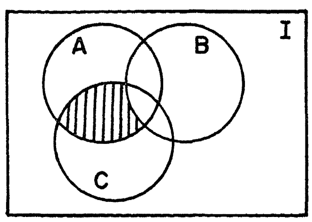
Answer twenty-five of the thirty questions in this part. Each correct answer will receive two credits. No partial credit will be allowed.

Directions (1–10): For *each question chosen,* in the space provided on the separate answer sheet, write the *numeral* preceding the expression that best completes the statement or answers the question.

1 Which is the negation of the statement "We don't have class only if the teacher is sick"?
(1) We do have class only if the teacher isn't sick.
(2) The teacher is sick and we do have class.
(3) We don't have class and the teacher isn't sick.
(4) If the teacher isn't sick, then we have class.
(5) We do have class and the teacher isn't sick.

2 If $p \to \sim r$ and $p \wedge \sim q$ are true statements, then
(1) $\sim q \to \sim r$
(2) $\sim r \to q$
(3) $\sim q \wedge r$
(4) $q \vee r$
(5) $\sim(q \to r)$

3 In the Venn diagram, *A, B,* and *C* are interiors of the circles lying within the rectangle *I*. The shaded area is represented by
(1) $A \cap (B \cap C)'$
(2) $(A \cup C) \cap B'$
(3) $(A \cap C) \cap B$
(4) $(B \cap C) \cap A'$
(5) $(A \cap C) \cap B'$



4 Which is the negation of the statement "Some equations are quadratic equations"?
(1) All equations are quadratic equations.
(2) No equations are quadratic equations.
(3) Some equations are not quadratic equations.
(4) At least one equation is not a quadratic equation.
(5) There exists an equation which is a quadratic equation.

5 Which set of numbers forms a group where the operation given is ordinary addition or ordinary multiplication?
(1) $\{-1,0,1\}$ under addition
(2) $\{-1,0,1\}$ under multiplication
(3) the set of even integers under addition and multiplication
(4) the set of complex numbers of the form $a + bi$, where a and b are real, under multiplication
(5) the set of odd integers under multiplication

6 Which of the following relations is *not* a function?
(1) $\{(0,0),(2,1),(3,2),(4,3)\}$
(2) $\{(x,y) \mid y^2 = x,\ x \text{ and } y \text{ are real}\}$
(3) $\{(x,y) \mid y = |x - 1|,\ x \text{ and } y \text{ are real}\}$
(4) $\{(x,y) \mid y = |x| + 1,\ x \text{ and } y \text{ are real}\}$
(5) $\{(x,y) \mid (y = \sqrt{x - 1}) \wedge (x \geqq 1),\ x \text{ and } y \text{ are real}\}$

7 Let f and g be functions defined for real x as follows: $f = \{(x,y) \mid y = x^2 - 3 \text{ and } x \neq \pm\sqrt{3}\}$, $g = \{(x,y) \mid y = 2x + 3 \text{ and } x \neq 1\}$. If the composite function $g(f)$ is formed, which of the following describes the domain of $g(f)$?
(1) all real numbers except $\pm\sqrt{3}$
(2) all real numbers except 1
(3) all real numbers except 1 and $\pm\sqrt{3}$
(4) all real numbers except ± 2 and $\pm\sqrt{3}$
(5) all real numbers except 1 and $\dfrac{-3 \pm\sqrt{3}}{2}$

8 If f and g are functions defined for real x as $f(x) = x - \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2 - 1}$, then the domain of the function which is their product, $f \cdot g$, is
(1) all real numbers
(2) all real numbers except 0
(3) all real numbers except 1 and -1
(4) all real numbers except 0, 1, and -1
(5) all real numbers except 0 and 1

9 Line *AB* is oblique to plane *m*. A plane containing *AB* is
(1) sometimes parallel to *m*
(2) sometimes perpendicular to *m*
(3) always parallel to *m*
(4) always perpendicular to *m*
(5) never perpendicular to *m*

10 Given two nonempty sets A and B, with $A \subset B$. It will always be true that
(1) $A \cup B = B$
(2) $A \cup B = A$
(3) $A \cup B = A \cap B$
(4) $A \cup B = \phi$
(5) $A \cap B = \phi$

11 Find the coordinates of the point of inflection of the graph of the function $y = 2x^3 - 3x^2 - 12x - 2$.

12 The set $G = \{e,a,r,l\}$ forms a group under the operation $*$, defined by the table:

$*$	e	a	r	l
e	e	a	r	l
a	a	e	l	r
r	r	l	a	e
l	l	r	e	a

What is the inverse of the element l?

13 Consider the function defined by $y = \dfrac{2 - x}{3 + x}$, where the domain of f is $\{x \mid (x \text{ is real}) \wedge (x \neq -3)\}$. What is the domain of $f^{-1}(x)$?

14 Find a value of x that satisfies the equation $\log_{100} (x^2 - 3x) = \frac{1}{2}$.

15 If $f(x) = \dfrac{3x^2 - 3}{x - 1}$ for all real x where $x \neq 1$, define $f(1)$ so that $f(x)$ will be continuous at $x = 1$.

16 Write the equation of a circle if the extremities of its diameter are the points where the line $2x + 3y - 12 = 0$ intersects the coordinate axes.

17 The equation of a certain parabola is $x^2 + 2x + 4y - 11 = 0$. What are the coordinates of its focus?

18 Find the rectangular coordinates for the point whose polar coordinates are $\left(2, \dfrac{2\pi}{3}\right)$.

19 Find the smallest positive integer which satisfies the congruence $3x - 2 \equiv x + 7, \pmod{11}$.

20 A box contains 5 red, 4 white, and 3 black balls. If 2 balls are drawn at random, without replacement, what is the probability that both are white?

21 Find the rational root of $3x^3 + 2x^2 + 3x + 2 = 0$.

22 The complex number $a + bi$ is represented by the ordered pair (a,b). Find the value of (a,b) if $\dfrac{(4,1)}{(a,b)} = (3,-2)$.

23 If a coin is tossed 3 times, what is the probability that the number of heads that occur is greater than the number of tails?

24 Express in polar form the root of $x^5 - 32 = 0$ which lies in the second quadrant.

25 Find the product of the roots of the equation $2x^5 + 6x^2 - x - 2 = 0$.

26 When $x^3 + kx + 4$ is divided by $x - 1$, the remainder is the same as when it is divided by $x + 2$. What is the value of k?

27 A spherical balloon is inflated with gas at the rate of 100 cubic feet per minute. Assuming that the gas pressure remains constant, how fast is the radius of the balloon increasing at the instant when the radius is 5 feet?

28 Find the equation of the line tangent to the parabola $y = 8x^2$ at the point $(\frac{1}{2},2)$.

29 Determine the limit of the sequence whose general term is $\dfrac{n^2 - 4}{3n^2 + 2n + 1}$, as n increases without bound.

30 Solve the inequality $x^2 - 3x > 10$.

Answers to the following questions are to be written on paper provided by the school.

Part II

Answer five questions from this part.

31 Prove by mathematical induction that, for all positive integers n, the expression $4^n - 1$ is divisible by 3. [10]

32 The function f is defined

$$f(x) = \begin{cases} 2x, \text{ when } x < -1 \\ 4, \text{ when } x = 1 \\ x + 5, \text{ when } x > 2 \\ 3, \text{ otherwise,} \end{cases}$$

where the domain of x is the set of real numbers.

a Draw the graph of the function for $-5 \leqq x \leqq 5$. Indicate missing points by the symbol o. [4]
b What is the range of $f(x)$? [2]
c For which value(s) of x is the function discontinuous? [3]
d Is the inverse of f a function? [Answer yes or no.] [1]

33 The rectangular coordinates of the vertices of a pyramid are A (0,0,0), B (4,0,0), C (4,6,0), D (0,6,0), and E (4,6,5).

a Write an equation of the trace in the xy-plane of the plane perpendicular to the xy-plane and passing through the points A and B. [2]
b Find an equation of the plane which contains the lateral face BCE. [2]
c Given that F is the midpoint of AD, find the length of EF. [2]
d Find the volume of pyramid $ABCDE$. [4]

34 Find all the roots of the equation $x^4 - 4x^3 - 5x^2 + 36x - 36 = 0$. [10]

35 A rectangular lot adjacent to a highway is to be enclosed by a fence. If the fencing costs \$2.50 per foot along the highway and \$1.50 per foot on the other sides, find the dimensions of the largest lot that can be fenced off for \$720. [10]

36 Using the definition of the derivative, find the derivative of the function $f(x) = \dfrac{2x + 5}{x - 3}$. [10]

37 Find an equation of a parabola whose axis is parallel to the x-axis, whose vertex is on the y-axis, and which passes through (2,4) and (8,—2). [10]

38 *a* Each time Irene and Barbara play a certain game, the probability of Irene winning is $\frac{2}{3}$ and the probability of a tie is 0. If they play 6 games, what is the probability that Barbara wins exactly 4. [5]

b If $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \cup B) = 0.5$, find

(1) $P(A \cap B)$ [3]
(2) $P(B \mid A)$ [2]

FOR TEACHERS ONLY

SCORING KEY

12X

EXAMINATION IN EXPERIMENTAL
TWELFTH YEAR MATHEMATICS

JUNE 1967

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Part I

Allow 50 credits, 2 credits for each of 25 of the following:

(1) 3

(2) 1

(3) 5

(4) 2

(5) 4

(6) 2

(7) 4

(8) 4

(9) 2

(10) 1

(11) $(\frac{1}{2}, -8\frac{1}{2})$

(12) r

(13) $\{x \mid x \text{ is real and } x \neq -1\}$

(14) 5 *or* —2

(15) 6

(16) $(x-3)^2+(y-2)^2=13$

(17) $(-1,2)$

(18) $(-1,\sqrt{3})$

(19) 10

(20) $\frac{1}{11}$

(21) $-\frac{2}{3}$

(22) $\left(\frac{10}{13}, \frac{11}{13}\right)$

(23) $\frac{1}{2}$

(24) $2\left(\cos\frac{4\pi}{5} + i \sin\frac{4\pi}{5}\right)$

(25) 1

(26) —3

(27) $\frac{1}{\pi}$ ft. per min.

(28) $y = 8x - 2$

(29) $\frac{1}{3}$

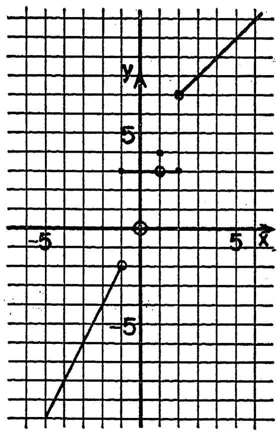
(30) $\{x \mid x < -2 \vee x > 5\}$

[OVER]

Part II

Answer five questions from this part.

(32) *a*



[4]

b $\{y \mid y < -2 \vee y = 3 \vee y = 1 \vee y > 7\}$ [2]

c —1, 1, 2 [3]

d No [1]

(33) *a* $\dfrac{x}{4} + \dfrac{y}{6} = 1$ [2]

b $x = 4$ [2]

c $5\sqrt{2}$ [2]

d 40 [4]

(34) 2, 2, 3, —3 [10]

(35) 120 ft. × 90 ft. [10]

(37) $y^2 - 2x - 4y + 4 = 0$ *or*
$y^2 - 18x - 20y + 100 = 0$ [10]

(38) *a* $\dfrac{20}{243}$ [5]

b (1) 0.2 [3]
(2) $\frac{1}{2}$ [2]