The University of the State of New York

THE STATE EDUCATION DEPARTMENT

EXAMINATION IN EXPERIMENTAL TWELFTH YEAR MATHEMATICS

June 1966

The last page of the booklet is the answer sheet, which is perforated. Fold the last page along the perforation and then, slowly and carefully, tear off the answer sheet. Now fill in the heading of your answer sheet. When you have finished the heading, you may begin the examination immediately.

Part I

Answer twenty-five of the thirty questions in this part. Each correct answer will receive two credits. No partial credit will be allowed.

Directions (1–15): For *each question chosen,* in the space provided on the separate answer sheet, write the *numeral* preceding the expression that best completes the statement or answers the question.

1 Which is the negation of the statement "All persons are selfish"?
(1) All persons are unselfish.
(2) Some person is unselfish.
(3) Some persons are selfish.
(4) No person is selfish.
(5) No person is unselfish.

2 Which statement is a tautology?
(1) p
(2) $\sim p$
(3) $p \wedge \sim p$
(4) $\sim(p \wedge \sim p)$
(5) $\sim(p \vee \sim p)$

3 If $\sim p \vee r$ is a true statement, then
(1) $r \rightarrow p$
(2) $\sim p \rightarrow r$
(3) $\sim r \rightarrow \sim p$
(4) $r \leftrightarrow p$
(5) $\sim r \rightarrow p$

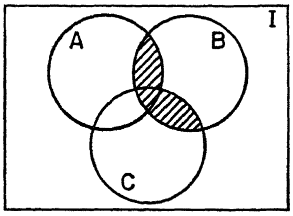
4 Which is the negation of "Vitamin A is essential or experiment 15 is not conclusive"?
(1) Vitamin A is not essential.
(2) Experiment 15 is conclusive.
(3) Vitamin A is not essential or experiment 15 is conclusive.
(4) Vitamin A is not essential and experiment 15 is conclusive.
(5) Vitamin A is not essential or experiment 15 is not conclusive.

5 In a multiple-choice test, five statements are given and labeled *a, b, c, d, e*. If at least two statements are tautologies, which answer to the following question *must* be true?
"Which of the statements *a, b, c, d, e* are tautologies?"
(1) *a, b, c, d,* and *e*
(2) *a, c,* and *e*, only
(3) *b, c,* and *e*, only
(4) *c, d,* and *e*, only
(5) *c* and *e*, only

6 If A is a set containing n elements, how many subsets does the cartesian product $A \times A$ have?
(1) 2^{2n}
(2) $2^{(n^2)}$
(3) n^2
(4) $2n$
(5) n

7 Let f be the function defined for real x as follows:
$$f(x) = 1 \text{ for } x \text{ irrational}$$
$$f(x) = 0 \text{ for } x \text{ rational}$$
Then the composite function $f(f(x))$ is the function whose value at x is
(1) 1 for all x
(2) 1 if x is irrational; 0 if x is rational
(3) 0 if x is irrational; 1 if x is rational
(4) 0 if x is rational; 2 if x is irrational
(5) 0 for all x

8 In the Venn diagram, A, B, and C are subsets of I. The shaded area is represented by
(1) $A \cap (B \cup C)$
(2) $A \cup (B \cap C)$
(3) $B \cap (A \cup C)$
(4) $B \cup (A \cap C)$
(5) $C \cap (A \cup B)$



9 If the universal set U is the set of real numbers and $A = \{x: |x| \geqq 2\}$ and $B = \{y: y < 2\}$, then
(1) $A \subset B$
(2) $B \subset A$
(3) $A' \cup B' = U$
(4) $A' \cap B = A'$
(5) $A \cap B = \phi$

10 For the function $y = f(x)$, which is defined and continuous over the closed interval $[a,b]$, the average rate of change of y with respect to x over the interval $[a,b]$ is given by

(1) $f'(x)$ evaluated at a

(2) $f'(x)$ evaluated at $\frac{a+b}{2}$

(3) $\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$ evaluated at $x = a$

(4) $\frac{f(b) - f(a)}{b - a}$

(5) $f''(x)$ evaluated at a

11 Let A represent the set of all zeros of polynomials with integral coefficients, and let B represent the set of all zeros of polynomials with rational coefficients. Which statement is true?

(1) $A \cup B$ equals the set of all real numbers.
(2) $A \cup B$ equals the set of all complex numbers.
(3) A is a proper subset of B.
(4) B is a proper subset of A.
(5) $A = B$.

12 If the numbers a and b are defined by
$a = 1.23451313$ and $b = 1.2345131\overline{3}\overline{1}\overline{3}\ldots$, then

(1) a is rational, b is irrational
(2) a is irrational, b is rational
(3) a and b are rational, and $\frac{a}{b}$ is irrational
(4) there exists a polynomial equation with integral coefficients having a for a root, but no such equation exists having b for a root
(5) $a, b, a+b, a-b, ab$, and $\frac{a}{b}$ are all rational

13 The lowest possible degree of a polynomial equation with real coefficients whose solution set has, as a subset, the set $\{1, i, 2+i, 2-i\}$ is

(1) 7 (3) 3 (5) 5
(2) 8 (4) 4

14 The number of solutions for x in the congruence
$\left(\sum_{i=1}^{5} ix\right) \equiv 1 \pmod{3}$ is

(1) 1 (3) 3 (5) infinite
(2) 2 (4) 0

15 The area of the right triangle of largest area which can be inscribed in a circle of radius 1 is

(1) 1 (3) $\sqrt{2}$ (5) $\pi\sqrt{2}$
(2) 2 (4) π

16 Solve for n: $3n^2 - 7n < 3 + 3(n-1)^2$

17 Evaluate exactly: $\log_{125}(.04)$

18 The function $g(x)$ is defined as follows:

$g(x) = \frac{2x^3+2}{x+1}$ for real x, $x \neq -1$, and
$g(-1) = c$.

If $g(x)$ is known to be a continuous function for all real x, then what is the value of c?

19 Find: $\lim_{x \to \infty} \frac{10^6x^6 + 10^5x^5 + 10^4x^4}{10^7x^7 + 10^6x^6 + 10^5x^5}$

20 The linear motion of a particle is given by its position s, in feet, at time t, in seconds, and obeys the exact functional relation expressed by the equation $s = t^4 - 3t^2 + 2$.

Find in feet per second per second the acceleration of the particle at time $t = 2$.

21 Find an equation of the curve each of whose points satisfies the condition that its distance from $(3,4)$ is equal to its distance from the line $x = -2$.

22 Find the positive abscissa of a point on the graph of $y = \frac{4}{3}x^3$ where the tangent line to the graph has an inclination of 45°.

23 What is the remainder free of x when $x^{24} - 5x^{21} + 3x^5 + 6$ is divided by $x + 1$?

24 Let the ordered pair (a,b), a and b rational, represent the complex number $a + bi$. This set of numbers, together with the usual complex number operations $+$ and $\times$, constitutes a field. Find the value of h if (g,h) is the multiplicative inverse of $(1,2)$.

25 Find in degrees the smallest positive amplitude m for which $[2(\cos m + i \sin m)]^{10}$ is a real number.

26 The digits 1, 2, 3, 4, and 5 are to be used to form 4-digit and 5-digit numbers that shall be odd numbers smaller than 40,000. If repeated use of the digits is allowed in forming these numbers, how many different numbers can be formed?

27 What is the probability that a positive integer x, selected at random, satisfies the congruence $x^2 \equiv 1 \pmod 5$?

28 The height of a right circular cone is h and the radius of its base is r. The cone is to be cut into two solids of equal volume by a plane parallel to the base. Find the altitude of the cutoff piece, which is a cone.

29 If $f(x)$ is the probability that exactly x heads have turned up during 6 tosses of a fair coin, find $f(3)$.

30 A right prism of altitude 6 has regular hexagonal bases, each of whose sides is 4. Find the length of the longest diagonal connecting a lower base vertex with an upper base vertex.

Answers to the following questions should be placed on paper supplied by the school.

Part II

Answer five questions from this part.

31 Find all the roots of the equation $5x^4 + 13x^3 + 9x^2 + 4x - 4 = 0$. [10]

32 *a* Prove by the use of truth tables that the following is a tautology:
$$[(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)] \wedge [(\sim q \rightarrow \sim p) \rightarrow (p \rightarrow q)]$$ [8]

b What does the fact that the above logical statement is a tautology prove about the relation of an implication to its contrapositive? [2]

33 Graph the six roots of the equation $x^6 - 64 = 0$, and write *one* of the nonreal roots in polar form *and* in rectangular form. [6,4]

34 Find the number of cubic units in the volume of the pyramid formed by the coordinate planes and the plane $60x + 36y + 45z = 180$. [10]

35 Prove by mathematical induction that, for all positive integers n, the expression $5^{2n} - 1$ is divisible by 12. [10]

36 An experiment consists of choosing an integer at random from among the natural numbers 1 through 40. We define the following events:

$P =$ "A prime number was chosen."
$C =$ "A composite number was chosen."
$I =$ "The integer 1 was chosen."
$M_n =$ "A multiple of n was chosen."

Note: The set of natural numbers is equal to the union of *the* three disjoint sets {1}, {positive prime integers}, {positive composite integers}.

Let $p(A) =$ probability of event A, etc.

For *our* experiment, find *each* of the following:

(1) $p(P)$ [2]
(2) $p(C)$ [2]
(3) $p(I \cup P \cup C)$ [2]
(4) $p(P \cap M_2)$ [2]
(5) $p(M_2 \cap M_3 \cap M_4 \cap M_6)$ [2]

37 Directly from the definition of a derivative, find the derivative of the function $f(x) = \sqrt{x^2 + 1}$. [10]

38 Consider the function $y = x + \dfrac{1}{x}$, defined on $\{x \mid x \text{ is real and } x \neq 0\}$.

a Find any maximum or minimum points of the function. State which are maximum and which are minimum. [4]

b In what regions is the function increasing? In what regions is it decreasing? [2,2]

c Sketch the function on a set of coordinate axes, using the information obtained in parts *a* and *b*. [2]

FOR TEACHERS ONLY

SCORING KEY

12X

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TWELFTH YEAR MATHEMATICS

JUNE 1966

Part I

Allow 50 credits, 2 credits for each of 25 of the following:

(1) 2

(2) 4

(3) 3

(4) 4

(5) 5

(6) 2

(7) 5

(8) 3

(9) 4

(10) 4

(11) 5

(12) 5

(13) 5

(14) 4

(15) 1

(16) $n > -6$

(17) $-\frac{2}{3}$

(18) 6

(19) 0

(20) 42

(21) $\sqrt{(x-3)^2 + (y-4)^2} = x + 2$ *or*
$y^2 - 8y - 10x + 21 = 0$

(22) $\frac{1}{2}$

(23) 9

(24) $-\frac{2}{5}$

(25) 18

(26) 1500

(27) $\frac{2}{5}$

(28) $\frac{h}{2}\sqrt[3]{4}$ *or* $\frac{h}{\sqrt[3]{2}}$

(29) $\frac{20}{64}$ *or* $\frac{5}{16}$

(30) 10

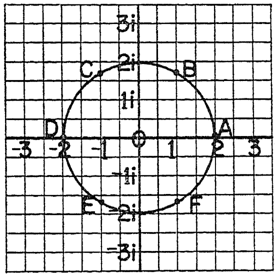
Part II

(31) $-2, \frac{2}{5}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ [10]

(32) *b* A statement such as "An implication and its contrapositive are equivalent" would be acceptable. [2]

[OVER]

(33)



[6]

$2(\cos 60° + i \sin 60°)$ and $1 + i\sqrt{3}$ [2, 2]
or $2(\cos 120° + i \sin 120°)$ and $-1 + i\sqrt{3}$
or $2(\cos 240° + i \sin 240°)$ and $-1 - i\sqrt{3}$
or $2(\cos 300° + i \sin 300°)$ and $1 - i\sqrt{3}$

(34) 10 [10]

(36) (1) $\frac{3}{10}$ [2]

(2) $\frac{27}{40}$ [2]

(3) 1 [2]

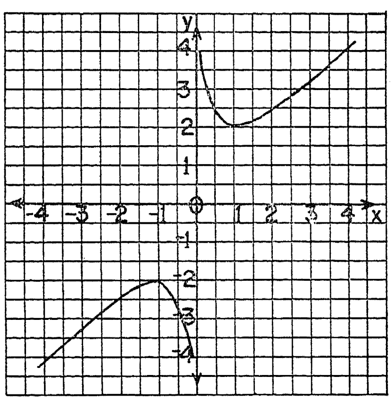
(4) $\frac{1}{40}$ [2]

(5) $\frac{3}{40}$ [2]

(38) *a* Maximum point $(-1,-2)$,
minimum point $(1,2)$ [4]

b Increasing;
$x < -1 \vee x > 1$ [2]
Decreasing;
$-1 < x < 0 \vee 0 < x < 1$ [2]

c



[2]