The University of the State of New York

THE STATE EDUCATION DEPARTMENT

EXAMINATION IN EXPERIMENTAL TWELFTH YEAR MATHEMATICS

June 1964

The last page of the booklet is the answer sheet, which is perforated. Fold the last page along the perforation and then, slowly and carefully, tear off the answer sheet. Now fill in the heading of your answer sheet. When you have finished the heading, you may begin the examination immediately.

Part I

Answer twenty-five of the thirty questions in this part. Each correct answer will receive two credits. No partial credit will be allowed.

Directions (1–12): For *each question chosen,* in the space provided on the separate answer sheet, write the *numeral* preceding the expression that best completes the statement or answers the question.

1 If h represents the statement "This coin can fall heads" and t represents the statement "This coin can fall tails," which symbolic expression represents a statement *equivalent* to "This coin can fall either heads or tails but not both heads and tails"?

(1) $(h \vee t) \vee (\sim h \vee \sim t)$
(2) $(h \vee \sim t) \wedge (t \vee \sim h)$
(3) $(h \wedge \sim t) \vee (t \wedge \sim h)$
(4) $(h \wedge \sim t) \wedge (t \wedge \sim h)$
(5) $(h \vee t) \wedge (\sim h \wedge \sim t)$

2 Which is *not* a tautology?

(1) $[p \wedge (p \rightarrow q)] \rightarrow q$
(2) $[(p \rightarrow q) \wedge (q \rightarrow p)] \leftrightarrow (p \leftrightarrow q)$
(3) $(p \wedge q) \leftrightarrow \sim(\sim p \vee \sim q)$
(4) $(p \rightarrow q) \rightarrow (\sim p \rightarrow \sim q)$
(5) $(p \wedge \sim p) \rightarrow q$

3 The contrapositive of the sentence $\sim p \rightarrow q$ is equivalent to

(1) $p \rightarrow \sim q$
(2) $q \rightarrow \sim p$
(3) $q \rightarrow p$
(4) $\sim p \rightarrow \sim q$
(5) $\sim q \rightarrow p$

4 Which system is an example of a group? [The addition and multiplication mentioned are the ordinary operations of arithmetic.]

(1) the set of nonnegative reals under addition
(2) the set of nonnegative integers under multiplication
(3) the set of nonnegative rationals under multiplication
(4) the set of positive reals under multiplication
(5) the set consisting of the four complex numbers which are the four fourth roots of unity under addition

5 If A and B are sets and A' and B' are their complements with respect to a universal set U and if $A \cap B' = \phi$ and $B \cap A' = \phi$, then it necessarily follows that

(1) $A = B$
(2) $(A \neq \phi) \wedge (B = \phi)$
(3) $(A = \phi) \vee (B' = \phi) \wedge (B = \phi) \vee (A' = \phi)$
(4) $(A = \phi) \wedge (B = \phi) \vee (A' = \phi) \wedge (B' = \phi)$
(5) $A \cap B = \phi$

6 If a set A consists of n elements and a set B consists of m elements, then the total number of *subsets* of the cartesian product $A \times B$ is

(1) mn
(2) $m + n$
(3) $2^{m \div n}$
(4) 2^{mn}
(5) $2^m + 2^n$

7 When simplified, the expression $3^{2 \log_3 x}$, where $x > 0$, reduces to

(1) $2x$
(2) 3^{x^2}
(3) 3^{2x}
(4) x^9
(5) x^2

8 The set $\{x \mid |x - L| < d\}$ is the same for all $d > 0$ and for all L, as

(1) $\{x \mid 0 < x < L + d\}$
(2) $\{x \mid L - d < x < L + d\}$
(3) $\{x \mid |L - d| < x < |L + d|\}$
(4) $\{x \mid |L - x| > d\}$
(5) $\{x \mid -d < |x - L|\}$

9 The solution set of $(x-2)(x-4)(x-6) > 0$ is
(1) $\{x \mid 2 < x < 4\} \cup \{x \mid x > 6\}$
(2) $\{x \mid 2 < x < 4\} \cap \{x \mid x > 6\}$
(3) $\{x \mid x < 2\} \cup \{x \mid 4 < x < 6\}$
(4) $\{x \mid x < 2\} \cap \{x \mid 4 < x < 6\}$
(5) $\{x \mid x < 2\} \cup \{x \mid x > 6\}$

10 The set of points in space 3 inches from a given line and 3 inches from a given point on this line is
(1) the empty set
(2) a set consisting of two points
(3) a set consisting of four points
(4) a set consisting of two circles
(5) a circle

11 If, $\forall_x \forall_y \forall_z$, x and y are irrational and z is rational, then which is irrational?
(1) $x + y$
(2) $z + x$
(3) $z \div x$
(4) xy
(5) zy

12 The sentence $\forall_x \; x + y = xz$ will be a true statement about real numbers for
(1) just one value of y and one value of z
(2) more than one value of y but for only one value of z
(3) more than one value of z but for only one value of y
(4) more than one value of y and for more than one value of z
(5) no values of y and z

13 The set of all numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers, form a field under addition and multiplication. Find the multiplicative inverse of $1 + 1\sqrt{2}$.

14 If the solution set of the congruence $4x + 2 \equiv 3 \pmod 7$ is to be the set $\{x \mid x = 2 + mk$, where k is any integer$\}$, what is the value of m?

15 If $g = \{(x,y) \mid y = x^2\}$ in which the domain is the set of all real numbers and $f = \{(0,7), (2,9), (4,11)\}$, write as a set of ordered pairs the composite function $g(f)$.

16 The function f is defined as

$$f = \{(x,y) \mid y = \frac{2x+1}{x-3}, \text{ where } x \neq 3\}.$$

Find the value of a so that the inverse of f will be

$$f^{-1} = \{(x,y) \mid y = \frac{3x+1}{x-a}, \text{ where } x \neq a\}.$$

17 Find the solution set of the inequality $2^{3x+3} > 2^{x-1}$.

18 Find the number of degrees in the angle between the lines whose equations are $y = 3x$ and $y = \frac{1}{2}x$.

19 Find the coordinates of the center of the circle which is the graph of the equation $y^2 = 18x - x^2$.

20 Write an equation for the set of points, the distance of each of which from the origin is twice its distance from the point (3,0).

21 Evaluate $\lim\limits_{n\to\infty} \dfrac{3n^2}{n^2 + 10{,}000n}$.

22 Write an equation for the line tangent to the ellipse $x^2 + 4y^2 = 25$ at the point (3,2) on the ellipse.

23 Write in *simplest* numerical form $\sum\limits_{k=1}^{6} \sin \dfrac{k\pi}{6}$.

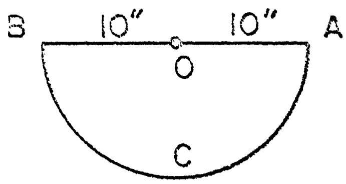
24 The motion of some object on the moon can be described approximately by the equation
$s = 200t - \frac{8}{3}t^2$
where s is measured in feet and t in seconds. From this equation find in feet per second per second the acceleration due to gravity on the moon.

25 Write a polynomial equation in x of lowest degree which has real coefficients and which has 0 and $1 + i$ as two of its roots.

26 The radius of a sphere is increasing at the rate of 2 inches per second. How fast is the volume increasing in cubic inches per second when the radius is 10 inches?

27 Write equations for the asymptotes to the curves whose equations are $xy = a$, where $a \neq 0$.

28 As shown in the accompanying figure, a cone is constructed from semicircle ACB by gluing radius OA to radius OB.



Find the number of inches in the radius of the base of the cone.

29 Find the constant remainder when $x^{27} - 3x^{15} + 2$ is divided by $x - 1$.

30 Write the repeating decimal 0.47474747... as a common fraction.

Part II

Answer five questions from this part.

31 Find all the roots of the equation
$3x^4 - 20x^3 + 41x^2 - 20x - 12 = 0$. [10]

32 *a* Using De Moivre's theorem, express all the roots of the equation $x^6 - 64 = 0$ in the field of complex numbers in both rectangular form (in the form $a + bi$) and in polar form (in the form $r(\cos\theta + i\sin\theta)$). [8]

b Find the sum of the roots of the equation in part *a*. [1]

c Find the product of the roots of this equation. [1]

33 Given an arithmetic progression in which the first term is a_1 and in which $a_{n+1} = a_n + d$ for all integers $n > 0$. Prove by mathematical induction that $\forall_n$ where n is a positive integer, $a_n = a_1 + (n - 1)d$. [10]

34 *a* Sketch, label and find the total surface area of the solid whose vertices are the points A (0,0,0,), B $(a,0,0)$, C $(a,b,0)$, D $(0,b,0)$, E $(0,0,c)$, F $(a,0,c)$, G (a,b,c) and H $(0,b,c)$. [4]

b Find the volume of the solid whose vertices are the points A,F,G,H given above. [2]

c Find the length of the line segment AG. [2]

d Write an equation for the plane determined by the points E,F,G. [1]

e Write the coordinates of the midpoint of the line segment AG. [1]

35 A farmer has 800 feet of fencing to enclose a rectangular plot of ground and to subdivide it into three rectangular cattle pens by means of two parallel partitions. What is the greatest area in square feet that he can put into pens? [10]

36 Find to the *nearest tenth* the positive root of the equation $4x^3 - 33x - 20 = 0$. [10]

37 A coin is tossed. If it falls heads, then one die is cast. If it falls tails, the coin is tossed again.

a Draw a tree diagram to show the set of outcomes of this experiment. [2]

b Find the probability of *each* of the following outcomes:

(1) $H\ 1$ [1]
(2) $H\ 1$ *or* $H\ 2$ [2]
(3) $H\ 1$ *and* $H\ 2$ [2]
(4) $T\ H$ [1]
(5) $H\ 1$ *or* $H\ 2$ *or* $H\ 3$ *or* $H\ 4$ *or* $H\ 5$ *or* $H\ 6$ *or* $T\ H$ *or* $T\ T$ [2]

38 A function f is defined as follows:

$$f(x) = \begin{cases} 2x, \text{ where } -1 \leq x < 1 \\ \dfrac{x^2 - 1}{x - 1}, \text{ where } 1 < x \leq 2 \\ \dfrac{4}{x}, \text{ where } 2 < x \leq 4 \end{cases}$$

a Sketch the graph of the function over the domain for which it is defined. Indicate missing points by the symbol o. [4]

b For which value(s) of x in the interval $-1 < x < 4$ is the function undefined? [2]

c For which value(s) of x in the interval $-1 < x < 4$ is the function discontinuous? [2]

d What is the range of the function? [1]

e Is the inverse of f a function? [Answer *yes* or *no*.] [1]

FOR TEACHERS ONLY

SCORING KEY

12X

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JUNE 1964

Part I

Allow 50 credits, 2 credits for each of 25 of the following:

(1) 3

(2) 4

(3) 5

(4) 4

(5) 1

(6) 4

(7) 5

(8) 2

(9) 1

(10) 5

(11) 2

(12) 1

(13) $-1 + 1\sqrt{2}$

(14) 7

(15) $g(f) = \{(0,49), (2,81), (4,121)\}$

(16) 2

(17) $\{x \mid x > -2\}$

(18) 45

(19) (9,0)

(20) $(x - 4)^2 + y^2 = 4$

or

$x^2 - 8x + 12 + y^2 = 0$

(21) 3

(22) $3x + 8y = 25$

(23) $2 + \sqrt{3}$

(24) $-\dfrac{16}{3}$ *or* $\dfrac{16}{3}$

(25) $x^3 - 2x^2 + 2x = 0$

(26) 800π

(27) $x = 0, y = 0$

(28) 5

(29) 0

(30) $\dfrac{47}{99}$

Part II

(31) $-\frac{1}{3}$, 2, 2, 3 [10]

(32) *a*

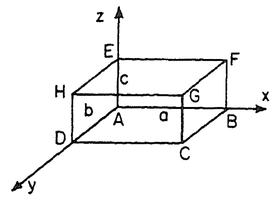
$2 + 0 \cdot i$	$2\ (\cos 0° + i \sin 0°)$
$1 + \sqrt{3}\,i$	$2\ (\cos 60° + i \sin 60°)$
$-1 + \sqrt{3}\,i$	$2\ (\cos 120° + i \sin 120°)$
$-2 + 0 \cdot i$	$2\ (\cos 180° + i \sin 180°)$
$-1 - \sqrt{3}\,i$	$2\ (\cos 240° + i \sin 240°)$
$1 - \sqrt{3}\,i$	$2\ (\cos 300° + i \sin 300°)$ [8]

b 0 [1]

c -64 [1]

[OVER]

(34) *a* $2ac + 2bc + 2ab$



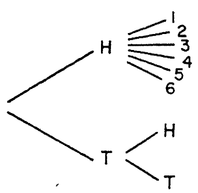
[4]

b $\frac{1}{6}\,abc$ [2]
c $\sqrt{a^2 + b^2 + c^2}$ [2]
d $z = c$ [1]
e $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ [1]

(35) 20,000 [10]

(36) 3.1 [10]

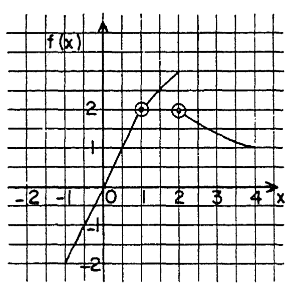
(37) *a*



[2]

b (1) $\frac{1}{12}$ [1]
(2) $\frac{1}{6}$ [2]
(3) 0 [2]
(4) $\frac{1}{4}$ [1]
(5) 1 [2]

(38) *a*



[4]

b 1 [2]
c 2 [2]
d from —2 to 3, including —2 and 3
but *not* including 2, *or* $\{f(x) \mid -2 \leq f(x) < 2\} \cup \{f(x) \mid 2 < f(x) \leq 3\}$ [1]
e No [1]